Assignment 2

Hand in no. 2, 5, 6, 8 and 9 by September 19, 2019.

1. Find the Fourier series of the function $|\sin x|$ on $[-\pi, \pi]$.

2. Show that
$$x^2 \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n},$$
for $x \in [0, 2\pi]$. Here we extend x^2 which is originally defined on $[0, 2\pi]$ to a 2π-periodic function on $\mathbb{R}$. Compare it with 4(a) in Assignment 1.

3. This is an optional problem.
   (a) Assume that the Fourier coefficients of a continuous, 2π-periodic function vanish identically. Show that this function must be the zero function. Hint: WLOG assume $f(0) > 0$. Use the relation
   $$\int_{-\pi}^{\pi} f(x)p(x)dx = 0,$$
   where $p(x)$ is a trigonometric polynomial of the form $(\varepsilon + \cos x)^k$ for some small ε and large $k > 0$.
   (b) Use the result in (a) to show that if the Fourier series of a continuous, 2π-periodic function converges uniformly, then it converges uniformly to the function itself.
   (c) Apply (b) to the Fourier expansion of x^2 to show that
   $$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

4. Let f be a complex valued 2π-periodic function whose derivative is again integrable on $[-\pi, \pi]$. Show that c_n and c_n', the Fourier coefficients of f and f' respectively, satisfies the relation $c_n' = inc_n, n \in \mathbb{Z}$. Do not do it formally. Use the definition of the integration of complex valued functions.

5. Let $C_{2\pi}^{\infty}$ be the class of all smooth 2π-periodic, complex-valued functions and $\mathcal{C}^{\infty}$ the class of all complex bisequences satisfying $c_n = \circ(n^{-k})$ as $n \to \pm\infty$ for every k. Show that the Fourier transform $f \mapsto \hat{f}$ is bijective from $C_{2\pi}^{\infty}$ to $\mathcal{C}^{\infty}$. Hint: You need to apply those theorems on uniform convergence in MATH2060.

6. Propose a definition for $\sqrt{d/dx}$. This operator should be a linear map which maps $C_{2\pi}^{\infty}$ to itself satisfying
$$\sqrt{\frac{d}{dx}}\sqrt{\frac{d}{dx}} f = \frac{d}{dx} f,$$
for all smooth, 2π-periodic f.

7. Let f be a continuous, 2π-periodic function and its primitive function be given by
$$F(x) = \int_0^x f(x)dx.$$
Show that F is 2π-periodic if and only if f has zero mean. In this case,
$$\hat{F}(n) = \frac{1}{in}\hat{f}(n), \quad \forall n \neq 0.$$

8. Let $\mathcal{C}'$ be the subspace of $\mathcal{C}$ consisting of all bisequences $\{c_n\}$ satisfying $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$.

   (a) For $f \in R[-\pi, \pi]$, show that

   $$\sum_{-\infty}^{\infty} |c_n|^2 \le \int_{-\pi}^{\pi} |f|^2 \ .$$

   (b) Deduce from (a) that the Fourier transform $f \mapsto \hat{f}(n)$ maps $R_{2\pi}$ into $\mathcal{C}'$.

   (c) Explain why the trigonometric series

   $$\sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha} \ , \quad \alpha \in (0, 1/2] \ ,$$

   is not the Fourier series of any function in $R_{2\pi}$.

9. Let f be a C^1-piecewise, continuous 2π-periodic function. In other words, there exist $-\pi = a_1 < a_2 < \cdots < a_N = \pi$ and C^1-functions f_j defined on $[a_j, a_{j+1}],\ j = 0, \cdots, N-1$ such that $f = f_j$ on (a_j, a_{j+1}). Show that its Fourier series converges uniformly to itself. Hint: Let $M = \max_j \{\sup |f_j'(x)| : \ x \in [a_j, a_{j+1}]\}$. Establish $|f(y) - f(x)| \le M|y - x|$ for all $x, y \in [-\pi, \pi]$.

10. Show that for a Lipschitiz continuous, 2π-periodic function, its Fourier coefficients satisfy

    $$|a_n| \le \frac{C\pi}{n}, \quad |b_n| \le \frac{C\pi}{n} \ ,$$

    for some constant C.